stichting
mathematisch
centrum

MC

AFDELING TOEGEPASTE WISKUNDE TW 167/77 JUNI
(DEPARTMENT OF APPLIED MATHEMATICS)

H.A. VAN DER MEER

OCTONIONS AND RELATED EXCEPTIONAL HOMOGENEOUS SPACES

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

AMS (MOS) subject classification scheme (1970): 17D05, 17C40 , 53C30 and 50D99.

Octonions and related exceptional homogeneous spaces

by

H.A. van der Meer

ABSTRACT

This paper describes several homogeneous spaces related to the algebra of octonions: the projective octonion plane $F_4/\mathrm{Spin}(9)$ and the spheres $S^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7)$, $S^7 = \mathrm{Spin}(7)/G_2$, $S^6 = G_2/SU(3)$. In order to make the exposition self-contained, the basic properties of the octonions and the Jordan algebra of 3×3 Hermitian matrices over the octonions are also derived.

The paper does not contain essentially new results, but it is intended as a rather elementary introduction to this subject.

KEYWORDS & PHRASES: *Homogeneous space, exceptional homogeneous space, composition algebra, non-associative algebra, octonions, exceptional Lie group, Jordan algebra.*

CONTENTS

Preface page 1

1. The problem of Hurwitz. 2

2. Arithmetics. 3

3. Triality. 7

4. The automorphism group of $\mathbb{O}$. 9

5. The exceptional Jordan algebra $\mathbb{J}_3(\mathbb{O})$. 15

6. Irreducible idempotents in $\mathbb{J}_3(\mathbb{O})$. 23

7. The projective octonion plane. 28

Appendix I. 31

Appendix II. 32

References. 33

PREFACE.

This paper is fitting in the research on "Special functions and group theory", which is part of the research program of the Department of Applied Mathematics.

The spherical functions on the homogeneous space $S^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7)$ (cf. TAKAHASHI [24], SMITH [21] and JOHNSON [17]) turn out to be orthogonal polynomials in two variables, belonging to a known class of special functions. In trying to understand this group theoretic interpretation it turned out that the algebraic preliminaries on octonions and Jordan algebras, as available in literature at the moment, are either rather inaccessible for analysts, or incomplete. Our intention in this exposé is to make things more clear for readers without much algebraic and Lie theoretic background.

Although our work is mainly aimed at assisting in the above mentioned investigations, it may be of some interest for people who are merely interested in the (introductory) theory of non-associative algebra.

The main subject of this paper is the algebra of octonions over the real numbers, which is described in detail and which is used to obtain several homogeneous spaces, in particular the one mentioned above and the Cayley elliptic plane $F_4/\mathrm{Spin}(9)$.

All homogeneous spaces considered are exceptional in the sense that they are not contained in any of the classical infinite sequences of homogeneous spaces, e.g. $S^{n-1} \approx SO(n)/SO(n-1)$.

Moreover, one might say that they form a complete set of exceptions in the classification of all transitive actions of compact connected simple Lie groups on simply-connected spaces, as was demonstrated in papers by A. BOREL [5],[6] and D. MONTGOMERY & H. SAMELSON [19].

Throughout, global methods are used, rather than infinitesimal methods.

The applications in the theory of special functions will be considered in a forthcoming report by T. Koornwinder and the author.

ACKNOWLEDGEMENT.

The help of T. Koornwinder, in particular his constructive criticism, was indispensable for writing this report.

1. THE PROBLEM OF HURWITZ.

(1.1) With a pair (V,N) we mean a vector space V over a field F of characteristic not two, and a mapping $N:V \to F$, that satisfies:

N1. $N(\alpha x) = \alpha^2 N(x) \qquad (\alpha \in F, x \in V)$

N2. $(.|.): V^2 \to F$, defined by
$(x|y) := \frac{1}{2}[N(x+y) - N(x) - N(y)]$
is bilinear.

(N is called a quadratic form on V).

We say that such a pair (V,N) is a composition algebra (c.a.) if, in addition to N1 and N2, it enjoys the following properties:

C1. V admits a bilinear composition, say xy, that is respected by N, i.e: $N(xy) = N(x)N(y) \qquad \forall x,y \in V$,

C2. the form $(.|.)$ is nondegenerate,

C3. V is finite dimensional and

C4. there is an identity element in V; 1.

<u>REMARK</u>. It is possible to prove the following assertion: V is a c.a. iff V is an alternative algebra (i.e. $x^2y = x(xy)$ and $yx^2 = (yx)x$ $\forall x,y \in V$) with an identity element and an involution $x \mapsto \overline{x}$, such that (i) $x+\overline{x} \in F$, (ii) $x\overline{x} =: N(x) \in F$, where F is the base field.

(1.2) <u>QUESTION</u>: (slightly simplified problem of Hurwitz). Given any base field F ($\mathrm{char}(F) \neq 2$), determine all composition algebras over F.

<u>ANSWER</u>: There are exactly four types of c.a's over F, of dimensions 1,2,4 and 8. They can be constructed from F by means of a certain doubling process, which is described below.

The proof of this statement (in [15]) is mainly based on two facts;
(1) this doubling can be reversed to "halving", which is applicable to any c.a. (and must eventually lead to the base field),
(2) starting with F we would lose alternativity in the fourth doubling (see remark above).

The doubling process looks as follows: let V be a c.a., μ a nonzero element of V, λ an element "outside" V, and consider $V' := V \oplus V\lambda$, equipped with the following multiplication:

(1) $(a+b\lambda)(c+d\lambda):=(ac+\mu\bar{d}b)+(da+b\bar{c})\lambda$;

the bar denotes the above mentioned involution in V.

HURWITZ [29] solved the problem himself, in 1898; other proofs are to be found in FREUDENTHAL [9], JORDAN etc. [18] (using representation theory), and JACOBSON [15].

(1.3) The purpose of this chapter is to show that so-called Cayley algebras (or octonion algebras) arise in a natural way in the solution of this problem: they are the composition algebras of dimension eight.
Until further notice we will restrict ourselves to the case $F = \mathbb{R}$, the field of real numbers.
μ has to be smaller than zero if we want to have division algebras, and it is most convenient to set $\mu = -1$.
Then:

dim 1 corresponds to $\mathbb{R}$
dim 2 corresponds to $\mathbb{C}$ (the complex numbers)
dim 4 corresponds to $\mathbb{H}$ (the quaternions)
dim 8 corresponds to $\mathbb{O}$ (the octonions).

REMARK: Octonions (or octaves) were first mentioned by HAMILTON [28] in 1848, after they had been discovered (?) earlier by Graves. Sometimes they are called: Graves-Cayley numbers.

2. ARITHMETHICS.

(2.1) We set $\mathbb{K}_0 = \mathbb{R}$, $\mathbb{K}_1 = \mathbb{C}$, $\mathbb{K}_2 = \mathbb{H}$, $\mathbb{K}_4 = \mathbb{O}$;
Choosing the symbols for λ accordingly, we have

$$\mathbb{K}_1 = \mathbb{K}_0 \oplus \mathbb{K}_0 e_1$$
$$\mathbb{K}_2 = \mathbb{K}_1 \oplus \mathbb{K}_1 e_2$$
$$\mathbb{K}_4 = \mathbb{K}_2 \oplus \mathbb{K}_2 e_4 .$$

From the multiplication rule (1.2.1) (with $\mu = -1$) one sees:

(1) $e_1^{\;2} = e_2^{\;2} = e_4^{\;2} = -1$

(2) $\mathbb{H}$ is not commutative (e.g. $e_1e_2 = -\,e_2e_1$)

(3) $\mathbb{O}$ is neither commutative nor associative
(e.g. $e_1(e_2e_4) = -\,(e_1e_2)e_4$).

In $\mathbb{K}_0$ we put $\overline{x} := x$, $N(x) := |x|^2 = x^2$, and inductively we give the following definitions:

(4) $\overline{x} := \overline{x}_1 - x_2e_i$ (for $x \in \mathbb{K}_i$, $x = x_1 + x_2e_i$, $x_1, x_2 \in \mathbb{K}_j$,
$(i,j) \in \{(1,0),(2,1),(4,2)\}$)

By induction: $\overline{\overline{x}} = x$ and $\overline{xy} = \overline{y}\,\overline{x}$.

(5) $N(x) = |x|^2 := x\overline{x} = \overline{x}x \quad (= |x_1|^2 + |x_2|^2)$

(6) $\mathrm{Re}(x) := \frac{1}{2}(x+\overline{x})$
(so $\mathrm{Re}(xy) = \mathrm{Re}(yx)$ and $\mathrm{Re}(x) = \mathrm{Re}(\overline{x})$)

(7) $(x|y) := \mathrm{Re}(x\overline{y}) = \mathrm{Re}(\overline{y}x)$
$(= \frac{1}{2}(|x+y|^2 - |x|^2 - |y|^2))$.

This defines a real inner product in $\mathbb{K}_i$.
It follows that

(8) $|xy| = |x|\,|y|$

(The proof of this identity is rather computational. In applying the induction process it is crucial that one starts with an associative algebra.)

(9) $x^{-1} := |x|^{-2}\,\overline{x}$ $(x \neq 0)$ is the two-sided inverse of x in $\mathbb{K}_i$.

(2.2) Next, we derive a number of identities, which are valid in $\mathbb{K}_i$ for $i = 0,1,2,4$, but mostly trivial for $i = 0,1,2$. The main purpose is finding substitutes for associativity in $\mathbb{K}_4 = \mathbb{O}$.
Let x,y,x',y',a,z be arbitrary elements of $\mathbb{K}_i$.
By (2.1.7) and (2.1.8) we have

(1) $(x|x)(y|y) = (xy|xy)$.

Linearization leads to

(2) $(xy'|x'y) + (xy|x'y') = 2(x|x')(y|y')$.

With the help of this identity we derive

$$\begin{aligned}
(ay'|y) - (y'|\overline{a}y) &= (ay'|y) - (y'|(2\mathrm{Re}(a)-a)y)\\
&= (ay'|y) + (y'|ay) - 2\mathrm{Re}(a)(y'|y)\\
\text{(taking } x = a,\ x' = 1) &= 2(a|1)(y'|y) - 2\mathrm{Re}(a)(y'|y)\\
&= 0.
\end{aligned}$$

Hence,

(3) $(ax|y) = (x|\overline{a}y)$ and $(xa|y) = (x|y\overline{a})$.

From $(x||a|^2 y - \overline{a}(ay)) = |a|^2 (x|y) - (ax|ay)$
$= 0$ (by (2))

and the nondegeneracy of $(.|.)$ it follows that

(4) $\overline{a}(ay) = |a|^2 y = (ya)\overline{a}$.

Substituting $\overline{a} = 2\mathrm{Re}(a)-a$ gives

(5) $a^2 y = a(ay)$ and $ya^2 = (ya)a$,

which are the defining relations for alternative algebras (cf. (1.1)). Let

$$A(x,y,z) := (xy)z - x(yz).$$

This expression is called the associator of x,y and z. Linearization of (4) and (5) yields

(6) $A(x,y,z) = \mathrm{sign}(\sigma)\, A(\sigma(x),\sigma(y),\sigma(z))$, $(\sigma \in S_3)$.

But then $A(a,x,\overline{a}) = A(a,x,a) = 0$, so

(7) $(ax)\overline{a} = a(x\overline{a})$ and $(ax)a = a(xa)$.

Finally, we will derive two important identities, due to Ruth Moufang:

(8) $a(xy)a = (ax)(ya)$

(9) $(axa)y = a(x(ay))$

(in subsequent sections referred to as the first and second Moufang identity, respectively.)

PROOF. $(a(xy)a|z) = (a(xy)|z\overline{a})$

(by (2)) $= 2(a|z)(xy|\overline{a}) - (|a|^2|z(xy))$.

$$((ax)(ya)|z) = (ax|z(\bar{a}\,\bar{y}))$$
$$(\text{by } (2)) = 2(a|z)(x|\bar{a}\,\bar{y}) - (|a|^2|(zx)y).$$

Since $(1|z(xy)) = (1|(zx)y)$, these expressions are equal and, with the nondegeneracy of $(.|.)$, (8) follows.

Furthermore, $(a(x(ay))|z) = (x(ay)|\bar{a}\,z)$
$$= (x|(\bar{a}\,z)(\bar{y}\,\bar{a}))$$
$$(\text{by } (8)) = (x|\bar{a}(z\bar{y})\bar{a})$$
$$= ((axa)y|z). \quad \square$$

(2.3) Summary:

(2.2.3) $(ax|y) = (x|\bar{a}y), (xa|y) = (x|y\bar{a})$

(2.2.4) $\bar{a}(ay) = |a|^2 y = (ya)\bar{a}$

(2.2.5) $a^2y = a(ay)$, $ya^2 = (ya)a$

(2.2.6) (actually a result from (2.2.6))

$$(xy)z + y(zx) = x(yz) + (yz)x$$
$$(xy)z + (xz)y = x(yz) + x(zy)$$
$$(xy)z + (yx)z = x(yz) + y(xz)$$

(2.2.7) $(ax)\bar{a} = a(x\bar{a}), a(xa) = (ax)a$

(2.2.8) $(ax)(ya) = a(xy)a$

(2.2.9) $(axa)y = a(x(ay))$.

(2.4) We can easily find orthonormal bases for $\mathbb{K}_i$ $(i = 1,2,4)$, over the real numbers, $\mathbb{K}_0$:
$(1,e_1)$ for $\mathbb{C}$,
$(1,e_1,e_2,e_1e_2)$ for $\mathbb{H}$, and
$(1,e_1,e_2,e_1e_2,e_4,e_1e_4,e_2e_4,(e_1e_2)e_4)$ for $\mathbb{O}$.
It is convenient to set $e_1e_2 = e_3$, $e_1e_4 = e_5$, $e_2e_4 = e_6$ and $e_3e_4 = e_7$.
(Note that for all i,j we have $e_ie_j = \pm e_r$ for some r).
An octonion x can thus be represented as
$x = x_0+x_1e_1+\dots\dots\dots+x_7e_7$, $x_i \in \mathbb{R}$.
Now $|x|^2 = x_0^2 + x_1^2+\dots\dots\dots+x_7^2$, $\bar{x} = x_0 - x_1e_1\dots\dots\dots-x_7e_7$,
$Re(x) = x_0$ and, if $y = y_0 + y_1e_1+\dots\dots\dots+y_7e_7$:
$(x|y) = x_0y_0 + x_1y_1+\dots\dots\dots+x_7y_7$.
We will conclude this section with some remarks about the structure of this basis.

(1) $e_i^2 = -1$, $\quad 1 \le i \le 7$

(2) $e_ie_j = -e_je_i$, $\quad 1 \le i \ne j \le 7$

(3) if $e_ie_j = e_r$ then $e_{\sigma(i)}e_{\sigma(j)} = \mathrm{sign}(\sigma)e_{\sigma(r)}$, $\quad (\sigma \in S_3)$

(4) $e_i(e_je_r) = -e_j(e_ie_r)$, $\quad 1 \le i \ne j \le 7$ & $1 \le r \le 7$

(5) $e_i(e_je_r) = -(e_ie_j)e_r$, $\quad 1 \le i \ne j \ne r \ne i \le 7$.

(1),(2) and (3) can be found by the multiplication rule (1.2.1), and combination of them yields (4) and (5).
A corollary is

(6) $A(x,y,z) = 0 \;\forall\, x, y \in \mathbb{O}$ implies $z \in \mathbb{R}$.

This will be used in the next section.

3. TRIALITY.

(3.1) The orthogonal group of the eight dimensional Euclidean space can be identified with that of $\mathbb{O}$, with respect to the bilinear form $(.|.)$, defined in the previous chapter. This group is denoted by $O(8)$, and its subgroup consisting of matrices X with $\det(X) = 1$ by $SO(8)$.

PRINCIPLE OF TRIALITY:
For each $T \in SO(8)$, *there are* $T_1, T_2 \in SO(8)$ *such that:*

(1) $T(x)T_1(y) = T_2(xy) \qquad \forall\, x, y \in \mathbb{O}$.

Moreover: (T_1,T_2) *being such a pair implies: the only other pair satisfying* (1) *is* $(-T_1,-T_2)$.

PROOF. We make use of two facts:
(i) each $T \in SO(8)$ is a product of an even number of reflections in seven dimensional subspaces. Such reflections have the form:
$S_a(x) = x - 2(x|a)a$, where $a \in \mathbb{O}$, $|a| = 1$, so $S_a(x) = -a\bar{x}a$
(by (2.1.6) and (2.1.7)).

(ii) If L_a denotes the left translation by a ($L_a(x) = ax$) and $|a| = 1$, then $L_a \in SO(8)$ (analogous for R_a and T_a; respectively the mappings: $x \longmapsto xa$ and $x \longmapsto axa$).

Suppose (T,T_1,T_2) and (T',T_1',T_2') satisfy (1), then (TT',T_1T_1',T_2T_2') does so, whence we only have to prove the existence of (T_1,T_2) for T being the product of two reflections, say S_aS_b. Starting from (1), T_1 and T_2 will be chosen along the way:

$$\begin{aligned} T_2(xy) &= S_aS_b(x)T_1(y) \\ &= [a(\bar{b}x\bar{b})a]\ T_1(y) \\ &= a[(\bar{b}x\bar{b})(aT_1(y))] \qquad (*) \end{aligned}$$

(the second Moufang identity is used; (2.2.9)).
A sensible choice for T_1 seems to be $L_{\bar{a}}L_b$:

$$\begin{aligned} (*) &= a[(\bar{b}x\bar{b})(by)] \\ &= a[\bar{b}(xy)] \text{ (again by (2.2.9))}, \end{aligned}$$

with the obvious conclusion: $T_2 = L_aL_{\bar{b}}$.
Regarding the uniqueness: suppose $T \in SO(8)$ admits two pairs; (T_1,T_2) and (T_1',T_2'), then the identity matrix I admits the pair $(T_1'T_1^{-1}, T_2'T_2^{-1})$. Now:

taking $x = 1$ leads to: $T_1'T_1^{-1} = T_2'T_2^{-1}$ = say T_3.

$xT_3(1) = T_3(x)$ implies $x(yT_3(1)) = (xy)T_3$, whence (by (2.4.6)) $T_3(1) \in \mathbb{R}$ and, with $|T_3(1)| = 1$: $T_3(1) = \pm 1$. Hence $T_3 = \pm I$ and $(T_1',T_2') = (\pm T_1,\pm T_2)$. □

<u>REMARK</u>. There is a geometrical interpretation of triality in the case that $\mathbb{O}$ is split (i.e. no division algebra). This, and the proof above can be found in VAN DER BLIJ & SPRINGER [4].

(3.2) If $T \in O(8)\backslash SO(8)$, (that is, if $\det T = -1$), then T is made up by an odd number of reflections. It is not difficult to show (with the help of the first Moufang identity) that T satisfies the so-called second kind of triality:

$$\exists T_1,T_2 \in O(8)\backslash SO(8) : T(y)T_1(x) = T_2(xy), \qquad \forall x,y\in\mathbb{O}.$$

<u>REMARK</u>. Since T can satisfy only one kind of triality at the time, we have here a criterion for the determinant of T.

(3.3) Let $\Delta := \{(T,T_1,T_2) \in (SO(8))^3 | T(x)T_1(y) = T_2(xy)\ \forall x,y\in\mathbb{O}\}$, and $\Delta := \{(T,T_1,T_2) \in \Delta | T(1) = 1\}$. ($T(1) = 1 \Leftrightarrow T_1 = T_2$). Identifying SO(7) with the subgroup of SO(8) consisting of matrices T with $T(1) = 1$, we find two continuous epimorphisms:

$$\Pi : \Delta \longrightarrow SO(8)$$

$$\Pi : \Delta' \longrightarrow SO(7),$$

defined by $\Pi(T,T_1,T_2) = T$ and $\Pi'(T,T_1,T_1) = T$.
Π and Π' have discrete kernels: $\ker(\Pi) = \ker(\Pi') = \{(I,I,I),(I,-I,-I)\}$; therefore they are 2-1 coverings.

To show that Δ and Δ' are pathwise connected, it obviously suffices to find an arc connecting (I,I,I) and $(I,-I,-I)$.

Let $T_a : \mathbb{O} \longrightarrow \mathbb{O}$ be defined by $T_a(x) = axa$, and $c_\varphi := \cos \pi\varphi + e_1 \sin \pi\varphi$; a norm-one octonion. Then the arc

$$\varphi \longmapsto (T_{c_\varphi}, L_{\bar{c}_\varphi}, L_{c_\varphi}) \qquad (\varphi \in [0,1])$$

has the required property.

We conclude that Δ *and* Δ' *are isomorphic with* Spin(8) *and* Spin(7), *respectively, where* Spin(n) *denotes the universal covering group of* SO(n).

4. THE AUTOMORPHISM GROUP OF $\mathbb{O}$.

(4.1) The automorphism group of $\mathbb{O}$ consists of those invertible $\mathbb{R}$-linear transformations of $\mathbb{O}$ which satisfy

(1) $\alpha(x)\alpha(y) = \alpha(xy) \qquad \forall x,y\in\mathbb{O}$.

Hence $\alpha(1) = 1$ and $\alpha(e_i)^2 = -1$, $1 \le i \le 7$, so $\mathrm{Re}(\alpha(e_i)) = 0$ which in its turn yields $\overline{\alpha(e_i)} = -\alpha(e_i)$, and, in general $\overline{\alpha(x)} = \alpha(\overline{x})$. Then

$$\forall x\in O \quad |\alpha(x)|^2 = \alpha(x)\overline{\alpha(x)} = \alpha(|x|^2) = |x|^2.$$

Conclusion: $\alpha \in O(7)$, *so* Aut $(\mathbb{O}) \subset O(7)$

($\alpha \in O(7)$ iff $\alpha \in O(8)$ and $\alpha(1) = 1$).

(4.2) Let $X := \{(a,b) \in \mathbb{O}^2 \mid (a|1) = (b|1) = (a|b) = 0$ and $|a| = |b| = 1\}$. $SO(7)$ is transitive on X (i.e. $\forall (a,b),(c,d)\in X \quad \exists T\in SO(7)$: $(Ta,Tb) = (c,d)$), and the stabilizer of $(e_1,e_2) \in X$ is $SO(5)$.
(i.e. $T \in SO(5)$ iff $(Te_1,Te_2) = (e_1,e_2)$).
Hence $X \approx SO(7)/SO(5)$ as a homogeneous space.
For $(a,b) \in X$ choose $c \in O$ such that $(c|ab) = (c|1) = (c|a) = (c|b) = 0$ and $|c| = 1$.
(Such an element exists). Then the linear transformation β, defined by $\beta(1) = 1$, $\beta(e_1) = a$, $\beta(e_2) = b$, $\beta(e_3) = ab$, $\beta(e_4) = c$, $\beta(e_5) = ac$, $\beta(e_6) = bc$, $\beta(e_7) = (ab)c$ is an automorphism, as can be seen from the multiplication rules for the basis elements, in a mostly trivial way. (One has to realize that for purely imaginary octonions x,y: $xy = -yx$ and $x^2 \in \mathbb{R}$.)

For this reason, the automorphism group of $\mathbb{O}$, which we will denote by G for a while, is also transitive on X.

(Note that each automorphism of $\mathbb{O}$ will act on $\{1,e_1,\dots\dots,e_7\}$ as β does, for some a,b,c.)

The stabilizer of (e_1,e_2) in G is the subgroup G_D, consisting of all automorphisms that leave $D := \langle e_1,e_2\rangle \subset \mathbb{O}$ pointwise fixed.
($\langle\dots\rangle :=$ subalgebra generated by ....).

Concerning G_D it can be observed:

(i) $\alpha \in G_D \Rightarrow \alpha$ is determined by $\alpha(e_4) \in D^\perp$, $|\alpha(e_4)| = 1$;

(ii) conversely, if $c \in D^\perp$, $|c| = 1$, an automorphism can be defined such that $\alpha|_D = id_D$ and $\alpha(e_4) = c$.

We have $D \approx D^\perp \approx \mathbb{H}$ $(D^\perp = \mathbb{H}e_4)$, so we could look upon G_D as the unit sphere in $\mathbb{H}$; S^3. (The action of G_D on $D^\perp$ equals the action of SU(2) on $\mathbb{H}$: $x \in D^\perp$, $\alpha(xe_4) = x(x_0e_4) = (x_0x)e_4$ (by (2.4.4) and (2.4.5)) for a certain $x_0 \in D$ with $|x_0| = 1$. Hence $\alpha \in G_D$ acts on $\mathbb{H}$ by left multiplication with a unit vector)
We conclude that (1) $X \approx {}^G/G_D$,

(2) *since* X *and* G_D ($\approx S^3$) *are (simply) connected,* G *is (simply) connected and*

(3) $\mathrm{Dim}(G) = \mathrm{Dim}(X) + \mathrm{Dim}(G_D) = 11+3 = 14$.

(The dimension of X can be found by counting parameters).
Those three facts point to a well-known fact in Lie theory: G is a compact real form of the exceptional simple Lie group G_2.

(This was first observed by CARTAN (1925), who did however not prove it. Complete demonstrations can be found in e.g. FREUDENTHAL [9] (calculation of the root-system of G) or SPRINGER [23] (on which paper the part above was inspired)).

Henceforth, we will write $G_2 = \mathrm{Aut}(\mathbb{O})$.

(4.3) $\mathbb{O}$ contains the field of complex numbers as a subalgebra: $\mathbb{C} = \{x+ye_1 | x,y \in \mathbb{R}\} \subset \mathbb{O}$. It is easily verified that $z(we_i) = (zw)e_i$ for $z,w \in \mathbb{C}$ and $i > 1$. Therefore, we can consider $\mathbb{O}$ as a left vector space over $\mathbb{C}$. As a basis we take $\{1,e_2,e_4,e_6\}$. Rules for multiplication in this space are found by means of those for the e_i's. They are ($z,w \in \mathbb{C}$; $i,j \in \{2,4,6\}$):

(i) $(ze_i)\, w = (z\bar{w})e_i$,

(2) $(ze_i)(we_j) = (\bar{z}\,\bar{w})(e_ie_j) \quad i \neq j$,

(3) $(ze_i)(we_i) = -z\overline{w}$,

(4) $z(we_i) = (zw)e_i$.

Let $(.|.)^{\mathbb{C}}$ denote the Hermitean inner product on $\mathbb{O}$ with respect to $\{1,e_2,e_4,e_6\}$. This form can be expressed in terms of the real one:

(5) $(x|y)^{\mathbb{C}} = (x|y) + (x|e_1y)e_1 =: Co(x\overline{y})$,

where $x \longmapsto Co(x)$ denotes the orthogonal projection of $\mathbb{O}$ on its subspace $\mathbb{R} \oplus \mathbb{R}e_1$.

Let $U(4)$ be defined as the group of unitary transformations of $\mathbb{O}$ with respect to $(.|.)^{\mathbb{C}}$, and let $U(3) = \{T \in U(4) | T(1) = 1\}$. From (5) it follows that $\{\alpha \in G_2 | \alpha(e_1) = e_1\} \subset U(3)$. The contents of (4.4) were suggested to the author by T. Koornwinder.

(4.4) Using the notation of (3.3) we have two groups Δ and Δ', which are isomorphic with Spin(8) and Spin(7), respectively. Let us identify $SO(n)$, for $n \leq 7$, with

$\{T \in SO(8) | T(1) = 1 \text{ and } T(e_i) = e_i, \text{ for } i = 1,2,\ldots,7-n\}$.

Then the group

$\{(T_1,T,T) \in \Delta' | T_1 \in SO(n)\}$

is isomorphic with Spin(n). We can (and will, in the following) identify Spin(n) (for $n \leq 7$) also with a subgroup of SO(8):

$\mathrm{Spin}(n) = \{T_1 \in SO(8) | \exists\, T \in SO(n): (T,T_1,T_1) \in \Delta'\}$.

<u>LEMMA 1</u>. $\mathrm{Spin}(6) = U(4) \cap \mathrm{Spin}(7)$.

<u>PROOF.</u> "$\subset$": For $T \in \mathrm{Spin}(6)$ we have $T_1(x)T(y) = T(xy)$, for a certain $T_1 \in SO(6)$ and all $x,y \in \mathbb{O}$.

Then $\quad T_1(x\bar{y})T(y) = |y|^2 T(x)$

$$T_1(x\bar{y})|T(y)|^2 = |y|^2 T(x)\overline{T(y)}$$

(1) $\quad T_1(x\bar{y}) = T(x)\overline{T(y)}.$

Further, we have $(x|y)^{\mathbb{C}} = Co(x\bar{y}) = Co(T_1(x\bar{y}))$

by (1) $\qquad = Co(T(x)\overline{T(y)})$

$\qquad = (T(x)|T(y))^{\mathbb{C}}$,

so $T \in U(4)$. Clearly $Spin(6) \subset Spin(7)$.

"$\supset$": $T \in U(4) \cap Spin(7)$ implies: $\exists\, T_1 \in SO(7)$ such that $T_1(x)T(y) = T(xy)$, $\quad \forall x,y \in \mathbb{O}$.

By applying (1) we find: $Co(T_1(e_1)) = Co(T(e_1)\overline{T(1)}) = Co(e_1) = e_1$. Since $|T_1(e_1)| = 1$: $\; T_1(e_1) = e_1$. Thus $T \in Spin(6)$. $\quad\square$

Let $SU(4) \subset U(4)$ be the subgroup that is composed of matrices with determinant one. (In general: $T \in U(4) \Rightarrow \det(T) = e^{i\varphi}$).

<u>REMARK</u>: We use two symbols to denote the same element of $\mathbb{O}$: $i = e_1$. This will, however, not be confusing.

<u>PROPOSITION 2</u>. $Spin(6) = SU(4)$.

<u>PROOF</u>. In view of the previous lemma it only has to be proved that $SU(4) = U(4) \cap Spin(7)$.

Let $T \in U(4)$, with eigenvalues $e^{i\varphi}$ ($\varphi = \varphi_1,\varphi_2,\varphi_3,\varphi_4$) and corresponding eigenvectors $a = a_1,a_2,a_3,a_4$. Then T is the product of four pairs of reflections (see appendix I):

$$S_{e^{\frac{1}{2}i\varphi}a} \cdot S_a, \quad \text{for} \quad (a,\varphi) = (a_1,\varphi_1),(a_2,\varphi_2),(a_3,\varphi_3),(a_4,\varphi_4).$$

(where the a_j's are normalized to: $|a_j| = 1$).

$(L_{e^{\frac{1}{2}i\varphi}a} L_{\bar{a}},\ R_{e^{\frac{1}{2}i\varphi}a} R_{\bar{a}},\ S_{e^{\frac{1}{2}i\varphi}a} S_a)$ is an element of Δ (by (2.2.8)).

Taking the product of four of such elements for $(a,\varphi) = (a_j,\varphi_j)$, $j = 1,2,3,4$; we obtain $(T_1,T_2,T_3) \in \Delta$ with $T = T_3$.

But $L_{e^{\frac{1}{2}i\varphi_a}} L_{\overline{a}}(1) = e^{\frac{1}{2}i\varphi}$, hence $T_1(1) = e^{\frac{1}{2}i(\varphi_1+\varphi_2+\varphi_3+\varphi_4)}$. We mention two possibilities: 1) $T_1(1) = 1$, then $T = T_3 \in \mathrm{Spin}(7)$, and 2) $T_1(1) = -1$, then $-T_1 \in SO(7)$, whence $(-T_1,-T_2,T_3) \in \Delta$ and still $T \in \mathrm{Spin}(7)$.

Conversely, $T \in \mathrm{Spin}(7)$ implies 1) or 2), so we have proved:
$T \in \mathrm{Spin}(7)$ iff $T_1(1) = \pm 1$ iff $e^{i(\varphi_1+\varphi_2+\varphi_3+\varphi_4)} = 1$ iff $T \in SU(4)$ (since $T_1(1)^2 = \det(T)$). □

(4.5) COROLLARY 1. *The stabilizer of* e_1 *in* G_2 *is* $SU(3)$.

PROOF. $\mathrm{Stab}(e_1) = G_2 \cap SO(6) = \mathrm{Spin}(6) \cap SO(6) = SU(4) \cap SO(6) = SU(3)$. □

The set of purely imaginary octonions of norm one:
$\{x \in \mathbb{O} \mid (x|1) = 0,\ |x| = 1\}$ is obviously homeomorphic with $S^6 \subset \mathbb{R}^7$; the six-dimensional unit sphere. Since G_2 is transitive on X (defined in (4.2)), it is also transitive on S^6. Hence we have

PROPOSITION 2. $S^6 \approx {}^{G_2}/SU(3)$.

Identifying the set of all norm one octonions with the seven-dimensional unit sphere S^7, we obtain

PROPOSITION 3. $S^7 \approx {}^{\mathrm{Spin}(7)}/G_2$.

PROOF. We have proved that $SU(4)$ is contained in $\mathrm{Spin}(7)$. Therefore, $SU(4)$ being transitive on S^7 implies that $\mathrm{Spin}(7)$ has the same property.

The stabilizer of $1 \in S^7$ in $\mathrm{Spin}(7)$ is:
$\{T \in \mathrm{Spin}(7) \subset SO(8) \mid T(1) = 1\}$. But, if in in a triple $(T_1,T,T) \in \Delta'$ we have $T_1(1) = T(1) = 1$, then $T_1 = T$ and $T \in G_2$ □
(cf. (6.5), remark 3).

5. THE EXCEPTIONAL JORDAN ALGEBRA $\mathbb{J}_3(\mathbb{O})$.

(5.1) An important class of algebras was introduced by P. JORDAN [30] in 1933. One year later, in their joint paper "On the algebraic generalization of the quantummechanical formalism" [18], JORDAN, VON NEUMANN and WIGNER presented a detailed description of these algebras; the "r-number systems" as they were called by these authors. Later they were named after Jordan.

General definition of a Jordan algebra :

any algebra with identity, whose multiplication satisfies:

(1) $\quad xy = yx$

(2) $x^2(xy) = x(x^2y)$

for all elements x,y.

We will now define a number of Jordan algebras, that are in a way representative for all Jordan algebras (cf. [1],[16] or [18]).

Let $\{\mathbb{K}_i \ (i = 0,1,2,4)\}$ be the composition algebras over $\mathbb{R}$ (cf.(2.1)).

We recall that a square matrix T over an algebra with involution $(x \longmapsto \overline{x})$ is selfadjoint (or Hermitian) if $\overline{T} = T^t$ (the conjugate and the transpose of T, respectively).

Consider the sets $\mathbb{S}_n(\mathbb{K}_i)$ of selfadjoint n×n-matrices over $\mathbb{K}_i$ $(n = 1,2,\ldots\ldots)$.

Instead of the usual matrix product (denoted by XY) that does not generally preserve selfadjointness, we provide $\mathbb{S}_n(\mathbb{K}_i)$ with the following product:

(3) $X \circ Y = \frac{1}{2}(XY+YX)$,

which is

1) commutative but

2) in general not associative and

3) preserves selfadjointness, whence $\mathbb{S}_n(\mathbb{K}_i)$ has become an algebra.

For i = 0,1,2 and all n, $\mathbb{S}_n(\mathbb{K}_i)$ is a Jordan algebra, denoted by $\mathbb{J}_n(\mathbb{K}_i)$. (This is easy to check).

Regarding i = 4, there is the following result:

$\mathbb{S}_n(\mathbb{O})$ is not a Jordan algebra if $n > 3$ (see e.g. JACOBSON [16] p. 126/127). The case n = 1 is trivial; $\mathbb{S}_2(\mathbb{O})$ and $\mathbb{S}_3(\mathbb{O})$, denoted by $\mathbb{J}_2(\mathbb{O})$ and $\mathbb{J}_3(\mathbb{O})$, respectively, are indeed Jordan algebras, as will be stated in a corollary of proposition (5.4).

The automorphism groups of $\mathbb{J}_n(\mathbb{K}_i)$ are:

SO(n) modulo its centre for i = 0

SU(n) modulo its centre for i = 1

Sp(n) modulo its centre for i = 2,

(the last group being that of all symplectic matrices).
For $\mathbb{J}_2(\mathbb{O})$ it is SO(9) (see (6.5.4)). Finally, for $\mathbb{J}_3(\mathbb{O})$ it can be proved (*) that the automorphism group is a compact, real form of the exceptional simple Lie group F_4. Henceforth we will write accordingly: $F_4 = \mathrm{Aut}\ (J_3(\mathbb{O}))$.
The algebras $\mathbb{J}_3(A)$, with $\mathbf{A}$ an eight dimensional composition algebra over any base field of characteristic not two, play an outstanding role in the classification of Jordan algebras, and are therefore called exceptional Jordan algebras (cf. [1],[18]).

(5.2) An element of $\mathbb{J}_3(\mathbb{O})$ looks as follows:

$$X = \begin{pmatrix} x_1 & c_3 & \overline{c}_2 \\ \overline{c}_3 & x_2 & c_1 \\ c_2 & \overline{c}_1 & x_3 \end{pmatrix}, \qquad \underline{x} = (x_1,x_2,x_3) \in \mathbb{R}^3,\ \underline{c} = (c_1,c_2,c_3) \in \mathbb{O}^3.$$

This is abbreviated by $X = X(\underline{x},\underline{c})$.
We distinguish six elements:

$$E_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad E_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_3 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad F_1^c := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & \overline{c} & 0 \end{pmatrix}$$

$$F_2^c := \begin{pmatrix} 0 & 0 & \overline{c} \\ 0 & 0 & 0 \\ c & 0 & 0 \end{pmatrix} \qquad F_3^c := \begin{pmatrix} 0 & c & 0 \\ \overline{c} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(*) CHEVALLEY & SCHAFER [8]; also [9] and [25].

Multiplication in $\mathbb{J}_3(\mathbb{O})$ can be described in terms of these six elements:

(1) $E_i \circ E_j = 0$ and $E_i \circ E_i = E_i$, $\quad 1 \le i \neq j \le 3$

$$\left.\begin{aligned} (2)\ & F_i^c \circ F_{i+1}^d = \tfrac{1}{2}F_{i+2}^{\overline{cd}}, \\ & F_i^c \circ F_i^d = (E_{i+1} + E_{i+2})(c|d) \end{aligned}\right\} \text{(indices mod 3)}$$

(3) $E_i \circ F_j^c = \tfrac{1}{2}F_j^c$, $E_i \circ F_i^c = 0$, $\quad 1 \le i \neq j \le 3$

(5.3) We will now examine some subgroups of F_4.
Let Spin(8) and Spin(7) denote the groups Δ and Δ' (defined in (3.3)) respectively, and G_2 the diagonal in Δ (i.e. the triples (T,T,T)). For $T \in SO(8)$, let T^* be the transformation $T^*(x) = \overline{T(\overline{x})}$, which is also an element of $SO(8)$.

PROPOSITION 1. (TAKAHASHI [24], p. 15):
Let $\alpha \in F_4$. *Then* $\alpha(E_1) = E_1$ *and* $\alpha(E_2) = E_2$ *iff there is a triple* $(T_1,T_2,T_3^*) \in$ Spin(8) *with*:

$$(1)\quad \alpha\begin{pmatrix} x_1 & c_3 & \overline{c}_2 \\ \overline{c}_3 & x_2 & c_1 \\ c_2 & \overline{c}_1 & x_3 \end{pmatrix} = \begin{pmatrix} x_1 & T_3(c_3) & \overline{T_2(c_2)} \\ \overline{T_3(c_3)} & x_2 & T_1(c_1) \\ T_2(c_2) & \overline{T_1(c_1)} & x_3 \end{pmatrix}, \quad \forall X = X(\underline{x},\underline{c}) \in \mathbb{J}_3(\mathbb{O})$$

PROOF. The identity element of $\mathbb{J}_3(\mathbb{O})$ is $I = E_1+E_2+E_3$. Since α is an automorphism: $\alpha(I) = I$, so $E_3 = I-E_1-E_2 = \alpha(I-E_1-E_2) = \alpha(E_3)$.
Suppose $\alpha(F_1^{c_1}) = Y(\underline{y},\underline{d})$, then $\alpha(E_2 \circ F_1^{c_1}) = E_2 \circ Y = \tfrac{1}{2}Y$ (5.2.3).

$$E_2 \circ Y = \begin{pmatrix} 0 & \tfrac{1}{2}d_3 & 0 \\ \tfrac{1}{2}\overline{d}_3 & y_2 & \tfrac{1}{2}d_1 \\ 0 & \tfrac{1}{2}\overline{d}_1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} y_1 & d_3 & \overline{d}_2 \\ \overline{d}_3 & y_2 & d_1 \\ d_2 & \overline{d}_1 & y_3 \end{pmatrix}.$$

Consequently $d_2 = y_1 = y_2 = y_3 = 0$. In the same way $E_3 \circ Y = \tfrac{1}{2}Y$ implies $d_3 = 0$. Hence $\alpha(F_1^{c_1}) = F_1^{c_1'}$; α induces only a transformation of c_1, which is orthogonal, as can be seen from

$$(E_2+E_3)|c_1'|^2 = \alpha(F_1^{c_1}) \circ \alpha(F_1^{c_1}) \qquad (5.2.2)$$
$$= \alpha[(E_2+E_3)|c_1|^2]$$
$$= (E_2+E_3)|c_1|^2.$$

The same argument is valid for $F_2^{c_2}$ and $F_3^{c_3}$. Accordingly we can write:

$$\alpha(F_i^{c_i}) = F_i^{T_i(c_i)}, \quad T_i \in O(8), \quad i = 1,2,3.$$

From (5.2.2), i = 1:

$$F_3^{T_3(\overline{cd})} = \alpha F_3^{\overline{cd}} = \alpha(F_1^{c} \circ F_2^{d}) = F_1^{T_1(c)} \circ F_2^{T_2(d)}$$
$$= F_3^{\overline{T_1(c)T_2(d)}}$$

Therefore, $T_1(c)\, T_2(d) = \overline{T_3(\overline{cd})} = T_3^*\,(cd)$ for all $c,d \in \mathbb{O}$; and $T_1,T_2,T_3 \in SO(8)$, (cf (3.2)); $(T_1,T_2,T_3^*) \in \mathrm{Spin}(8)$.

Regarding the converse: if for a triple (T_1,T_2,T_3^*), α is defined as in (1), it is easy to check, with the rules in (5.2), that $\alpha \in F_4$. □

Identifying Spin(8) with the automorphisms (1), we emphasize two consequences of the proposition 1.

COROLLARY 2: *Let* $\alpha \in \mathrm{Spin}(8) \subset F_4$. *Then*: $\alpha \in \mathrm{Spin}(7)$ *iff* $\alpha(F_1^{1}) = F_1^{1}$.

COROLLARY 3: *For* $\alpha \in \mathrm{Spin}(7) \subset F_4$: $\alpha \in G_2$ *iff* $\alpha(F_2^{1}) = F_2^{1}$.

The proofs are trivial.

(5.4) PROPOSITION 1: (FREUDENTHAL [9] & [10])
Each element of $\mathbb{J}_3(\mathbb{O})$ *can be brought to diagonal form by an element of* F_4.

Moreover, the coefficients of this diagonal matrix are unique up to permutations.

If $\mathbb{J}_3(\mathbb{O})$ is divided into equivalence classes:

(1) $[X] := \{\alpha(X) | \alpha \in F_4\} = \mathrm{Orbit}_{F_4}(X)$,

we can thus characterize these classes by unordered triples $(\lambda_1,\lambda_2,\lambda_3) \in \mathbb{R}^3$.

LEMMA 2. *Let* $T \in SU(3)$, *and define*

$$T_* : \mathbb{J}_3(\mathbb{O}) \longrightarrow \mathbb{J}_3(\mathbb{O})$$

by: (2) $T_*(X) := (TX)\overline{T}^t$. *Then* $T_* \in F_4$.

PROOF. T can be replaced by a product of three different kinds of matrices:

(i) elements of SO(3), (ii) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\varphi} & 0 \\ 0 & 0 & e^{-i\varphi} \end{pmatrix} \in SU(3)$

and (iii) $\begin{pmatrix} e^{i\varphi} & 0 & 0 \\ 0 & e^{-i\varphi} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SU(3)$

(for a proof, see appendix II). The reader will find no difficulties in proving that the lemma is correct for a T of type (i), (ii) or (iii). Further, we have (by (4.3.1/4)): $(T_1T_2)_* = (T_1)_*(T_2)_*$ for $T_1,T_2 \in SU(3)$. Now the proposition follows easily. □

REMARK. If T is of type (ii) or (iii), we have $T_* \in \mathrm{Spin}(8)$.

Proof of the proposition (5.4.1):

Let $X = X(\underline{x},\underline{c}) \in \mathbb{J}_3(\mathbb{O})$ and assume that $c_i \neq 0$, $i = 1,2,3$ (if this is false, there is only less work to be done). In a number of steps X will be transformed to a diagonal matrix, merely by applying elements of Spin(8) and automorphisms of type T_*.

Step 1: Use $\alpha_1 = (L_{\frac{\bar{c}_1}{|c_1|}}, R_{\frac{\bar{c}_1}{|c_1|}}, T_{\frac{\bar{c}_1}{|c_1|}}) \in \mathrm{Spin}(8)$ to obtain

$$X_1 := \alpha_1(X) = \begin{pmatrix} x_1 & c_3' & \bar{c}_2' \\ \bar{c}_3' & x_2 & |c_1| \\ c_2' & |c_1| & x_3 \end{pmatrix}.$$

Step 2: Find a $T \in SO(2)$ such that

$$T \begin{pmatrix} x_2 & |c_1| \\ |c_1| & x_3 \end{pmatrix} T^{-1} = \begin{pmatrix} x_2' & 0 \\ 0 & x_3' \end{pmatrix}.$$ Identify T with

$\begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix} \in SO(3)$. Then

$$X_2 = T_*(X_1) = \begin{pmatrix} x_1 & c_3'' & \bar{c}_2'' \\ \bar{c}_3'' & x_2' & 0 \\ c_2'' & 0 & x_3' \end{pmatrix}.$$

Step 3: With $\alpha_2 \in \mathrm{Spin}(8)$ we make c_2'' real (like in step 1), obtaining

$$X_3 := \alpha_2(X_2) = \begin{pmatrix} x_1 & c_3''' & |c_2''| \\ \bar{c}_3''' & x_2' & 0 \\ |c_2''| & 0 & x_3' \end{pmatrix}.$$

Step 4: We know that G_2 is transitive on spheres of purely imaginary octonions. Hence, there is a $\alpha_3 \in G_2$, $\alpha_3(c_3''') = \mathrm{Re}(c_3''') + |c_3''' - \mathrm{Re}(c_3''')|e_1$, and a corresponding element of F_4, which we will call also α_3. Thus:

$$X_4 := \alpha_3(X_3) = \begin{pmatrix} x_1 & z & |c_2''| \\ \overline{z} & x_2' & 0 \\ |c_2''| & 0 & x_3' \end{pmatrix} \quad \text{with } z \in \mathbb{C}.$$

<u>Step 5</u>: The Hermitian complex matrix X_4 can be transformed to diagonal form by a T_*, with $T \in SU(3)$ (see e.g. CHEVALLY [7] p.12/13).

<u>Proof of the uniqueness of this diagonal matrix</u>:

Suppose we have $\alpha_X(X) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$,

where α_x is the combination of all steps above. The trace of X has not changed during this process, hence

(3) $\sum_{i=1}^{3} \lambda_i = tr(\alpha_X(X)) = Tr(X)$,

(4) $\sum_{i=1}^{3} \lambda_i^2 = tr(\alpha_X(X)^2) = tr(\alpha_X(X^2)) = tr(X^2)$,

and, since $\alpha_X(X \circ X^2) = \alpha_X(X^2 \circ X)$ and α_X is 1-1, we have

$$X \circ X^2 = X^2 \circ X =: X^3, \qquad \text{so}$$

(5) $\sum_{i=1}^{3} \lambda_i^3 = tr(\alpha_X(X^3)) = tr(X^3)$.

The λ_i's are determined up to permutations by (3),(4) and (5), and these permutations can easily be accomplished by T_*'s.
E.g:

for $T = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \in SO(3)$, T_* interchanges λ_2 and λ_3. □

<u>COROLLARY 2</u>. $\mathbb{J}_3(\mathbb{O})$ *and* $\mathbb{J}_2(\mathbb{O})$ *are Jordan algebras.*

<u>PROOF</u>. For $X \in \mathbb{J}_3(\mathbb{O})$ we denote the diagonalizing automorphism by α_X (α_X is not uniquely determined by X). If $Y \in J_3(0)$, clearly

$\alpha_X(X \circ (X^2 \circ Y)) = \alpha_X(X^2 \circ (X \circ Y))$,

which yields the result, since α_X is injective. As concerns $\mathbb{J}_2(0)$; this is in an obvious way a subalgebra of $\mathbb{J}_3(\mathbb{O})$ (cf. (6.5)). □

<u>REMARK</u>. $\mathbb{J}_3(\mathbb{O})$ possesses a real inner product, defined by $(X|Y) := \mathrm{tr}(X \circ Y)$ which is invariant for F_4 by proposition (5.4.1). If a symmetric trilinear form is defined by $(X|Y|Z) := (X \circ Y\ Z)$, it can be proved that F_4 is the subgroup of all linear transformations of $\mathbb{J}_3(\mathbb{O})$, consisting of those that leave $(X|Y)$ and $(X|Y|Z)$ invariant (cf. CHEVALLY & SCHAFER [8]).

(5.5) Suppose that [X] is characterized by $(\lambda_1,\lambda_2,\lambda_3)$. The characteristic equation of $\alpha_X(X)$ is

$$\prod_{i=1}^{3} (\lambda_i - \lambda) = 0 \quad \text{or}$$

$$(1)\quad -\lambda^3 + (\lambda_1+\lambda_2+\lambda_3)\lambda^2 - (\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)\lambda + \lambda_1\lambda_2\lambda_3 = 0.$$

But $\lambda_1+\lambda_2+\lambda_3 = \mathrm{tr}(\alpha_X(X)) = \mathrm{tr}(X)$,

$\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1 = \frac{1}{2}((\mathrm{tr}(X))^2 - \mathrm{tr}(X^2))$ and

$\lambda_1\lambda_2\lambda_3 = \frac{1}{3}\,\mathrm{tr}(X^3) - \frac{1}{2}\,\mathrm{tr}(X^2)\mathrm{tr}(X) + \frac{1}{6}\,(\mathrm{tr}(X))^3.$

Substituting this in (1) and writing out the traces explicitly in the coefficients of $X = X(\underline{x},\underline{c})$ yields

$$(2)\quad -\lambda^3 + \lambda^2(x_1+x_2+x_3) + \lambda(|c_1|^2+|c_2|^2+|c_3|^2 - x_1x_2-x_2x_3-x_3x_1) +$$

$$+(x_1x_2x_3-x_1|c_1|^2-x_2|c_2|^2-x_3|c_3|^2+2\mathrm{Re}(c_1c_2c_3)) = 0.$$

This equation can be considered as a characteristic equation for elements of $\mathbb{J}_3(\mathbb{O})$.

For $\lambda = 0$ we obtain a generalized definition of the notion of a determinant:

$$\det(X) := \frac{1}{3}\,\mathrm{tr}(X^3) - \frac{1}{2}\,\mathrm{tr}(X^2)\mathrm{tr}(X) + \frac{1}{6}\,(\mathrm{tr}(X))^3.$$

6. IRREDUCIBLE IDEMPOTENTS IN $\mathbb{J}_3(\mathbb{O})$.

(6.1) $X \in \mathbb{J}_3(\mathbb{O})$ is said to be an irreducible idempotent if the following two conditions are satisfied.

(i) $X^2 = X$.

(ii) $X = X_1+X_2$ with $X_1^2 = X_1$, $X_2^2 = X_2$ and $X_1 \circ X_2 = 0$ implies: $X_1 = 0$ or $X_1 = X$.

Let $\mathbb{P}$ denote the set of all irreducible idempotents in $\mathbb{J}_3(\mathbb{O})$.

<u>LEMMA 1</u>. *For* $X = X(\underline{x},\underline{c})$ *the following assertions are equivalent:*

<u>$\mathbb{P}$1</u>: $X \in \mathbb{P}$.

<u>$\mathbb{P}$2</u>: $\mathrm{tr}(X) = 1$ and $x_ic_i = \overline{c_{i+1}c_{i+2}}$ (i = 1,2,3, indices mod. 3) with $|c_i|^2 = x_{i+1}x_{i+2}$.

<u>$\mathbb{P}$3</u>: $X \in [E_1](= [E_2] = [E_3])$ (cf. prop. (5.4.1)).

<u>$\mathbb{P}$4</u>: $\mathrm{tr}(X) = \mathrm{tr}(X^2) = \mathrm{tr}(X^3) = 1$.

<u>PROOF</u>. First, it should be mentioned that $\mathbb{P}$2 is equivalent with

<u>$\mathbb{P}$2'</u>: $\mathrm{tr}(X) = 1$ and $X^2 = X$, which is easy to verify.

Then, all implications follow from a consideration of the eigenvalues of an irreducible idempotent:

If $X \in \mathbb{J}_3(\mathbb{O})$ is an idempotent, we have

$X = X^2 = X^3$, so $\alpha_X(X) = \alpha_X(X^2) = \alpha_X(X^3)$, which implies, if $[X] \sim (\lambda_1,\lambda_2,\lambda_3)$ (cf. (5.4.1)),

$\lambda_i = \lambda_i^2 = \lambda_i^3$ for $i = 1,2,3$. Therefore, $\lambda_i = 0$ or $\lambda_i = 1$.

When X is moreover irreducible, $\alpha_X(X)$ must be so too, which leaves the only possibilities:

$\lambda_i = 1$ and $\lambda_{i+1} = \lambda_{i+2} = 0$, where $i = 1,2$ or 3. $\square$

(6.2) Let the stabilizer of E_1 in F_4 be denoted by S. For $X = X(\underline{x},\underline{c})$ and $Y = Y(\underline{y},\underline{d})$ being elements of $\mathbb{P}$, we have the following

<u>LEMMA 1.</u> $\exists\, \alpha \in S : \alpha(X) = Y$ *iff* $x_1 = y_1$.

<u>PROOF.</u> " $\Rightarrow$ ": $X = x_1E_1+x_2E_2+x_3E_3+F_1^{c_1}+F_2^{c_2}+F_3^{c_3}$.

$E_1 \circ X = x_1E_1+\frac{1}{2}(F_2^{c_2}+F_3^{c_3})$. Apply α to obtain

(*) $x_1E_1 + \frac{1}{2}\alpha(F_2^{c_2}+F_3^{c_3}) = E_1 \circ \alpha(X) = E_1 \circ Y = y_1E_1+\frac{1}{2}F_1^{d_1}+\frac{1}{2}F_2^{d_2}$

Suppose Z = one of $\alpha F_2^{c_2}$ and $\alpha F_3^{c_3}$. Then $E_1 \circ Z = \frac{1}{2}Z$ implies that the coefficient of E_1 in Z must be zero. Thus $\alpha F_2^{c_2}$ and $\alpha F_3^{c_3}$ do not contribute to the coefficient of E_1 in Y. With (*) it follows that $x_1 = y_1$.

" $\Leftarrow$ ": With the first two steps of the proof of (5.4.1) we can transform X into a matrix

$$\alpha(X) = \begin{pmatrix} x_1 & c_3' & \overline{c}_2' \\ \overline{c}_3' & x_2' & 0 \\ c_2' & 0 & x_3' \end{pmatrix},$$

by a certain $\alpha \in$ s. But $\alpha(X) \in \mathbb{P}$, so (by $\mathbb{P}2$) $c_1 = 0$ implies

$x_2' \cdot x_3' = 0$.

case one: $x_2' = x_3' = 0 \Rightarrow \alpha(X) = E_1$, whence, since $\alpha \in S$ and α is 1-1, $X = E_1 = Y$.

case two: say $x_2' \neq 0$ and $x_3' = 0$ (without loss of generality), then $c_2' = 0$ and

$$\alpha'(\alpha(X)) = \begin{pmatrix} x_1 & |c_3'| & 0 \\ |c_3'| & 1-x_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for a certain $\alpha' \in \mathrm{Spin}(8) \subset S$.

If we do the same for Y, we get the same matrix, since $x_1 = y_1$ and $|c_3'|^2 = x_1(1-x_1) = y_1(1-y_1) = |d_3'|^2$. □

PROPOSITION 2. *The orbits of* S *in* $\mathbb{P}$ *are characterized by a number:* $0 \le x_1 \le 1$, *which is the coefficient of* E_1 *in all elements of an orbit.*

PROOF. $X = X^2 \Rightarrow x_1 = x_1^2 + |c_2|^2 + |c_3|^2$ $(X = X(\underline{x},\underline{c}))$, so $0 \le x_1 \le 1$. Now the proposition is merely a corollary of the lemma (6.2.1). □

(6.3) LEMMA 1. $\mathrm{Orb}_S(E_2) \approx S^8 \subset \mathbb{R}^9$.

PROOF. $\mathrm{Orb}_S(E_2) \stackrel{(6.2.2)}{=\!=} \{X(\underline{x},\underline{c}) \in \mathbb{P} \mid x_1 = 0\}$. But $x_1 = 0$ implies $c_2 = c_3 = 0$ (cf. $\mathbb{P}2$ (6.1)). Hence, $\mathrm{Orb}_S(E_2)$ is composed of the matrices which have the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-x & c \\ 0 & \bar{c} & x \end{pmatrix} \quad \text{with } |c|^2 = x(1-x) \text{ or, rewritten : } |2c|^2 + (2x-1)^2 = 1. \qquad \square$$

PROPOSITION 2. *The stabilizer of* E_1 *in* F_4 *is isomorphic to* Spin(9), *the universal covering group of* SO(9).

PROOF. From the lemma it follows that S is transitive on S^8. If we look for the stabilizer of E_2 in S, we find (by (5.3))

$$\mathrm{Spin}(8) = (S \cap \mathrm{Stab}_{F_4}(E_2)).$$

Hence $S^8 \approx {}^S/\mathrm{Spin}(8)$; in particular:

S is connected and $\mathrm{Dim}(S) = 8 + \mathrm{Dim}(\mathrm{Spin}(8)) = 8 + \mathrm{Dim}(SO(8)) = \mathrm{Dim}(SO(9))$.

The action of S on S^8 is obviously the same as that of $O(9)$. Consequently, there is a continuous homomorphism $\varphi: S \longrightarrow O(9)$, representing this action.
If $\alpha \in S$ induces the identity on S^8, we have

$$\alpha(E_i) = E_i, \quad i = 1,2,3 \text{ and } \forall c \in \mathbb{O}: \quad \alpha(F_1^c) = F_1^c.$$

Hence $\alpha = (T_1,T_2,T_3) \in \mathrm{Spin}(8)$, with $T_1 = \mathrm{id}$. Thus, the kernel of φ consists of (I,I,I) and $(I,-I,-I)$. Furthermore:

(i) The image of φ is lying in $SO(9)$ since S is connected, and
(ii) is even equal to $SO(9)$, since $\mathrm{Dim}(S) = \mathrm{Dim}(SO(9))$ and since the kernel of φ is discrete. □

(6.4) Henceforth we will write $\mathrm{Spin}(9) = \mathrm{Stab}_{F_4}(E_1)$

<u>LEMMA 1</u>. *Let* $X_0 = X_0(\underline{x},\underline{c}) \in \mathbb{P}$ *such that* $0 < x_1 < 1$. *Then*

$$\mathrm{Orb}_{\mathrm{Spin}(9)}(X_0) \approx S^{15}.$$

<u>PROOF</u>.

Let $Y = x_1 \begin{pmatrix} 1 & d_3 & \overline{d}_2 \\ \overline{d}_3 & y_2 & d_1 \\ d_2 & \overline{d}_1 & y_3 \end{pmatrix} \in \mathbb{J}_3(\mathbb{O})$.

Then, by (6.1.3) $\mathbb{P}$2, we have:
$Y \in \mathbb{P}$ (and thus, by (6.2.1), $\in \mathrm{Orb}_{\mathrm{Spin}(9)}(X_0)$) iff

$d_1 = \overline{d_2 d_3}$, $y_2 = |d_3|^2$, $y_3 = |d_2|^2$ and $|d_2|^2 + |d_3|^2 = \frac{1}{x_1} - 1$. □

<u>PROPOSITION 2</u>. $S^{15} \approx {}^{\mathrm{Spin}(9)}/\mathrm{Spin}(7)$.

<u>PROOF</u>.

The stabilizer of $x_1 \begin{pmatrix} 1 & 0 & \sqrt{\frac{1-x_1}{x_1}} \\ 0 & 0 & 0 \\ \sqrt{\frac{1-x_1}{x_1}} & 0 & \frac{1-x_1}{x_1} \end{pmatrix} \in S^{15}$

in Spin(9) has to be the invariance group of E_1, E_3 and F_1^1, which equals Spin(7) (cf (5.3)). □

(6.5) LEMMA 1. *Let* $X, Y \in \mathbb{P}$. *Then*

$X \circ Y = \theta$ *iff* $\exists \beta_{XY} \in F_4$: $\beta_{XY}(X) = E_1$ *and* $\beta_{XY}(Y) = E_2$.

PROOF.
"$\Leftarrow$": $0 = E_1 \circ E_2 = \beta_{XY}(X \circ Y)$.
"$\Rightarrow$": From $E_1 \circ \alpha_X(Y) = 0$ it follows that

$$\alpha_X(Y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & y & d \\ 0 & \bar{d} & 1-y \end{pmatrix} \in \mathbb{P}. \text{ But then:}$$

$\alpha_X(Y) \in \mathrm{Orb}_{\mathrm{Spin}(9)}(E_2)$, so there is a $\gamma_Y \in$ Spin(9) with $\gamma_Y(Y) = E_2$. Thus $\beta_{XY} = \gamma_Y \alpha_X$. □

COROLLARY 2. *Let* X, Y, Z *be three non-zero elements of* $\mathbb{P}$. *Then*

$X \circ Y = Y \circ Z = Z \circ X = 0$ *implies* $X + Y + Z = I$.

PROOF. $\beta_{XY}(Z) = E_3$, as can be seen easily. □

In general:

$[\lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3] = \{\lambda_1 X + \lambda_2 Y + \lambda_3 Z \mid X, Y, Z \text{ as above}\}$

REMARK 1. By abuse of language we spoke about F_4 as being the group of automorphisms of $\mathbb{J}_3(\mathbb{O})$ restricted to $\mathbb{P}$. But this rectriction is obviously injective, so we are justified.

REMARK 2. It should be clear from the foregoing that the automorphism group of $\mathbb{J}_2(\mathbb{O})$ is Spin(9) modulo $\{(I,I,I),(I,-I,-I)\} \subset$ Spin(8), where $\mathbb{J}_2(\mathbb{O})$ is identified with all matrices

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & x_2 & c_1 \\ 0 & \bar{c}_1 & x_3 \end{pmatrix} \qquad \text{in } \mathbb{J}_3(\mathbb{O}).$$

This group is equal to SO(9), of course.

REMARK 3. Another proof of cor. (4.6.1):

$$S^7 \approx \{X = \frac{1}{3}\begin{pmatrix} 1 & c & c \\ \bar{c} & 1 & 1 \\ \bar{c} & 1 & 1 \end{pmatrix} \in \mathbb{P}\}$$

Spin(7) is transitive on this set, when considered as the stabilizer of $F_1^{\,1}$ in Spin(8) (cf (5.3.1)).

The stabilizer of $\frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ in

Spin(7) is $\{(T,T_1,T_1) \in \text{Spin}(7) \mid T_1(1) = 1\} = G_2$.

Hence $S^7 \approx {}^{\text{Spin}(7)}/G_2$. □

7. THE PROJECTIVE OCTONION PLANE.

(7.1) To start with, we will show some properties of $\mathbb{P}$, which will make proposition (7.2.1) more plausible.

LEMMA 1. *There is a continuous epimorphism:*

$$\Phi\colon (\text{Spin}(9))^3 \longrightarrow F_4.$$

PROOF. Say $(\text{Spin}(9))^3 = \text{Stab}(E_1) \times \text{Stab}(E_2) \times \text{Stab}(E_1) \subset F_4^{\,3}$ and let Φ be defined by

$$\Phi(\alpha_1,\alpha_2,\alpha_3) := \alpha_1\alpha_2\alpha_3.$$

We will see how any $\alpha \in F_4$ can be factorized in this manner.

Suppose $\alpha(E_1) = X(\underline{x},\underline{c})$. There are $\alpha_1 \in \mathrm{Stab}(E_1)$ and $\alpha_2 \in \mathrm{Stab}(E_2)$, such that:

$$E_1 \xrightarrow{\alpha} X \xrightarrow{\alpha_1} \begin{pmatrix} x_1 & 0 & \bar{c}_2' \\ 0 & 0 & 0 \\ c_2' & 0 & 1-x_1 \end{pmatrix} \xrightarrow{\alpha_2} E_1$$

(cf. (6.2.1): this lemma is also valid for $\mathrm{Stab}(E_2)$).
Thus $\alpha_2\alpha_1\alpha \in \mathrm{Stab}(E_1)$, say $\alpha_2\alpha_1\alpha = \alpha_3$. But then
$\alpha = \alpha_1^{-1}\alpha_2^{-1}\alpha_3 \in (\mathrm{Spin}(9))^3$. □

COROLLARY 2. F_4 *is compact and connected, and so are all of its orbits in* $\mathbb{J}_3(\mathbb{O})$.

(The first half of this statement is one of the things we took for granted in the previous section; though we did not use it).

LEMMA 3. $\mathrm{Dim}(\mathbb{P}) = 16$.

PROOF. $\mathrm{Dim}(\mathbb{P}) = \mathrm{Dim}(F_4) - \mathrm{Dim}(\mathrm{Spin}(9)) = 52 - 36 = 16$. □

(7.2) We recall a global (not complete) definition of a projective plane:

"An aggregate of two families: one being composed of points and one of lines. In this set an incidence relation between members of different families is defined, satisfying:

(i) A line is incident with at least three points, and

(ii) two different lines (points) are incident with exactly one point (line)."

If a projective plane over the octonions is to be constructed, conventional methods break down on the lack of associativity. (This is the case, for example, with the method of decrease of dimension). The following construction was carried out in detail by FREUDENTHAL [9].

DEFINITION. Let the set of points P and the set of lines L both be copies of $\mathbb{P}$, and let an incidence relation be defined as

$X \in P$ and $Y \in L$ are incident iff $X \circ Y = 0$.

PROPOSITION 1. *Being the "union" of* P *and* L, $\mathbb{P}$ *is provided with the structure of a projective plane (over* $\mathbb{O}$*).*

For complete demonstrations the reader should consult e.g. FREUDENTHAL [9], TITS [25] (this last author has a remarkable geometric approach) or to SPRINGER [22] (for an extensive and purely algebraic (general) presentation).

We will prove here only a part of the proposition.

- *Two different lines (points) are incident with at most one point (line).*

PROOF. Let $X,Y \in L(P)$ and $Z_1,Z_2 \in P(L)$, assuming that

(i) $X \neq Y$ (ii) $X \circ Z_i = Y \circ Z_i = 0$, $i = 1,2$.

Using the type of automorphism of which we proved the existence in (6.5.1) we obtain:

$\beta_{XZ_1}(Y \circ Z_1) = \beta_{XZ_1}(Y) \circ E_2 = 0$. Hence, $\beta_{XZ_1}(Y)$ has the form:

$$\begin{pmatrix} y & 0 & \bar{c} \\ 0 & 0 & 0 \\ c & 0 & 1-y \end{pmatrix}.$$

Furthermore, $\beta_{XZ_1}(X \circ Z_2) = E_1 \circ \beta_{XZ_1}(Z_2)$ so

$$\beta_{XZ_1}(Z_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & x & d \\ 0 & \bar{d} & 1-x \end{pmatrix} \text{ for some } x,d.$$

$$\beta_{XZ_1}(Y \circ Z_2) = \tfrac{1}{2} \begin{pmatrix} 0 & \overline{dc} & (1-x)\bar{c} \\ dc & 0 & (1-y)d \\ (1-x)c & (1-y)\bar{d} & 2(1-y)(1-x) \end{pmatrix},$$

and this must be the zero matrix.

If $y = 1$ then $\beta_{XZ_1}(Y) = E_1$ (since it is an element of $\mathbb{P}$), which implies $X = Y$ (β_{XZ_1} is injective). The only other possibility is $x = 1$, but then $\beta_{XZ_1}(Z_2) = E_2$, so $Z_1 = Z_2$. □

REMARK. The projective plane over $\mathbb{O}$ is also related to the exceptional Lie groups E_6, E_7 and E_8. We mention a few references considering these matters:

E_6 : FREUDENTHAL [9], TITS [26], CHEVALLEY & SCHAFER [8].
E_7 & E_8: FREUDENTHAL [11], TITS [26].
(There are, of course, many more)
For a survey, see FREUDENTHAL [12], or (very concise) VAN DER BLIJ [2].

APPENDIX I.

LEMMA. *A unitary transformation is a product of reflections.*

PROOF. $T \in U(n)$ implies that there is an Hermitian orthonormal basis of eigenvectors for $\mathbb{C}^n$, say $(v_1, \ldots, v_n)$, with

(1) $$T(v_k) = e^{i\varphi_k} v_k \qquad 1 \le k \le n.$$

We can write T in the form

$$T = \prod_{j=1}^{n} T_j,$$

were T_j is defined by

(2) $$T_j(v_j) = e^{i\varphi_j} v_j \quad \text{and} \quad T_j(v_k) = v_k, \qquad j \neq k.$$

Consider $\mathbb{C}^n$ as a real vector space, $\mathbb{R}^{2n}$, with inner product $(x|y)^{\mathbb{R}} := \mathrm{Re}(x|y)^{\mathbb{C}}$. The set of vectors $(v_1, iv_1, v_2, \ldots, iv_n)$ forms an orthonormal basis of $\mathbb{R}^{2n}$.

For T_j we find, in this language:

$$T_j(v_j) = v_j \cos\varphi_j + (iv_j)\sin\varphi_j,$$

$$T_j(iv_j) = -v_j \sin\varphi_j + (iv_j)\cos\varphi_j,$$

$$T_j(v_k) = v_k \quad \text{and} \quad T_j(iv_k) = iv_k \quad \text{if } j \neq k.$$

A reflection in the Euclidean space $\mathbb{R}^n$ is known to be defined for any $a \in \mathbb{R}^n$ with $|a| = 1$ by

$$S_a(x) = x - 2(x|a)^{\mathbb{R}} a.$$

Applying the pair $S_{e^{\frac{1}{2}i\varphi_j}v_j}\ S_{v_j}$ to our basis $(v_1, iv_1, iv_2, \ldots, iv_n)$ leads to the conclusion that its action is equal to the action of T_j. □

APPENDIX II.

DEFINITIONS:

$D(n)$:= set of complex, diagonal $n\times n$-matrices
$UD(n)$:= $U(n) \cap D(n)$
$SUD(n)$:= $SU(n) \cap D(n)$.

PROPOSITION. *For every* $T \in U(n)$, *there can be found* $T_1, T_2 \in O(n)$ *and* $A \in UD(n)$, *such that*

$$T = T_1\ A\ T_2.$$

PROOF. (by T. Koornwinder)
(T^t := transpose of T, T^* := conjugate of T^t)
Let $T \in U(n)$. Then $TT^t \in U(n)$, and

$$(TT^t)^* = (T^t)^*\ T^* = \overline{T}\ \overline{T}^t = \overline{T\ T^t}.$$

Hence, $TT^t = C+iD$, with C and D real and $CD = DC$ (since, in $U(n)$: $T^* = T^{-1}$).

Moreover, we have $C = C^t$ and $D = D^t$. A commuting pair of square, real, symmetric matrices can be diagonalized simultaneously:

$$\exists\ T_1 \in SO(n):\ T_1^{-1}\ C\ T_1,\ T_1^{-1}\ D\ T_1 \in D(n).$$

But then: $T_1^{-1}(TT^t)T_1 \in UD(n)$.

Choose $A \in UD(n)$ with $A^2 = T_1^{-1}(TT^t)T_1$.

Let $T_2 := A^{-1}T_1^{-1}T \in U(n)$.

Since $T_2T_2{}^t = A^{-1}T_1{}^{-1}TT^tT_1A^{-1}$

$$= A^{-1}T_1{}^{-1}(T_1A^2T_1{}^{-1})T_1A^{-1}$$

$$= I,$$

it follows that $T_2 \in O(n)$. Now we have T_1, T_2 and A satisfying the conditions of the proposition. □

COROLLARY (≈ LEMMA (5.4.2)): $\forall\ T \in SU(n)\ \ \exists\ T_1, T_2 \in SO(n)$ *and* $\exists\ A \in SUD(n)$, *with:*

$$T = T_1\ A\ T_2.$$

REFERENCES

The most important sources for the present paper are those marked by an asterisk.

References [27] to [31] are added, merely for the sake of historical interest.

[1] A.A. ALBERT ed., *Studies in modern algebra,* Mathematical Society of America (1963).

[2] F. VAN DER BLIJ, *History of the octaves,* Simon Stevin 34 (1961) p. 106-125.

[3] F. VAN DER BLIJ & T.A. SPRINGER, *Arithmetics of octaves and the group* G_2, Nederl. Akad. Wetensch. Proc. A 63 (1959) p. 406-418.

[4] F. VAN DER BLIJ & T.A. SPRINGER, *Octaves and triality,* Nieuw Arch. Wisk. (3) 8 (1960) p. 158-169.

[5] A. BOREL, *Some remarks about Lie groups transitive on spheres and tori,* Bull. Amer. Math. Soc. 55 (1949) p. 580-587.

[6] A. BOREL, *Le plan projective des octaves et les sphères comme espaces homogènes,* C.R. Sci. Paris 230 (1950) p. 1378-1380.

[7]* C. CHEVALLEY, *Theory of Lie groups,* Princeton U.P. (1946).

[8] C. CHEVALLEY & R.D. SCHAFER, *The exceptional Lie algebras* F_4 *and* E_6, Proc. Nat. Acad. Sci. U.S.A. 36 (1950) p. 137-141.

[9]* H. FREUDENTHAL, *Oktaven, Ausnahmegruppen und Oktavengeometrie*, Utrecht (1951, revised edition 1960).

[10]* H. FREUDENTHAL, *Zur ebenen Oktavengeometrie*, Nederl. Akad. Wetensch. Proc. A 56 (1953) p. 195-200.

[11] H. FREUDENTHAL, *Beziehungen der* E_7 *und* E_8 *zur Oktavenebene*, Nederl. Akad. Wetensch. Proc. A 57,58,62 (1954-55-59) (nine parts).

[12] H. FREUDENTHAL, *Lie groups in the foundations of geometry*, Advances in Math. 1 (1964) p. 145-190.

[13] F. GÜRSEY, *Exceptional groups and elementary particles*, In: Group theoretic methods in physics, T. Janssen and M. Boon eds., Lect. Notes in Physics 50 , Springer-Verlag, (1976).

[14] G. HIRSCH, *La géométrie projective et la topologie des espaces fibrés*, Coll. C.N.R.S., Top. Alg. Paris (1949) p. 35-42.

[15]* N. JACOBSON, *Composition algebras and their automorphisms*, Rend. Arc. Mat. Palermo, ser. II part VII (1958) p. 55-80.

[16] N. JACOBSON, *Structure and representations of Jordan algebras*, Amer. Math. Soc. coll. publ. 39 (1968).

[17] K.D. JOHNSON, *Composition series and intertwining operators for the spherical principle series II*, Trans. Amer. Math. Soc. 215 (1976) p. 269-283.

[18] P. JORDAN, J. VON NEUMANN & E. WIGNER, *On an algebraic generalization of the quantummechanical formalism*, Ann. of Math. 35 (1934) p. 29-64, or: Coll. Works J. von Neumann vol. II, p. 408-444, Pergamon press (1961).

[19] D. MONTGOMERY and H. SAMELSON, *Transformation groups of spheres*, Ann. of Math. 44 (1943) p. 454-470.

[20] R.D. SCHAFER, *An introduction to nonassociative algebras*, Academic Press (1966).

[21] R.T. SMITH, *The spherical representations of groups transitive on* S^n, Indiana Univ. Math. J. 24 (1974) p. 307-325.

[22] T.A. SPRINGER, *The projective octave plane I & II*, Nederl. Akad. Wetensch. Proc. A 63 (1960) p. 74-101.

[23]* T.A. SPRINGER, *Oktaven, Jordan-Algebren und Ausnahmegruppen*, Lecture Notes, Göttingen (1963).

[24]* R. TAKAHASHI, *Quélques résultats sur l'analyse harmonique dans l'espace symétrique non compact de rang 1 du type exceptionnel*, Preprint (1976).

[25] J. TITS, *Le plan projective des octaves et les groupes de Lie exceptionnels*, Acad. Roy. Belg. Bull. Cl. Sci. 39 (1953) p. 309-329.

[26] J. TITS, *Le plan projective des octaves et les groupes exceptionnels* E_6 *et* E_7, Ibidem 40 (1954) p. 29-40.

[27] B. ECKMANN, *Gruppentheoretischer Beweis des Satzes von Hurwitz-Radon etc.* Comment. Math. Helv. 15 (1942) p. 358-366.

[28] W.R. HAMILTON, *Note respecting the research of J.T. Graves*, Trans. Roy. Irish Acad. 21 (1848) p. 338-341.

[29] A. HURWITZ, *Ueber die Komposition der quadratische Formen von beliebig vielen Variabeln*, Nachr. Gött. (1898) p. 306-316 = Gött. Abh. II p. 565.

[30] P. JORDAN, *Ueber Verallgemeinerungsmöglichkeiten des Formalismus der Quantenmechanik*, Nachr. Gött. (1933) p. 209-214.

[31] P. JORDAN, *Ueber eine nicht-desarguessche ebene projektive Geometrie*, Abh. Math. Sem. Hamburg 16 (1949) p. 74-76.

Finally, we wish to mention a survey article, which could not be implicated in the preparation of this paper, since it was published this year ('77):

[32] J.R. FAULKNER & J.C. FERRAR, *Exceptional Lie algebras and related algebraic and geometric structures*, Bull. London Math. Soc. 9 (1977) p. 1-35.